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Stabilization of Arbitrary Switched Nonlinear Fractional Order Dynamical Systems: Application to Francis Hydro-Turbine Governing System

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This paper is a theoretical and practical study on the stabilization of fractional order Lipschitz nonlinear systems under arbitrary switching. The investigated system is a generalization of both switched and fractional order dynamical systems. Firstly, a switched frequency distributed model is introduced as an equivalent for the system. Subsequently, a sufficient condition is obtained for the stabilizability of the system based on the Lyapunov approach. Finally, the results are extended to synthesis mode-dependent state feedback controller for the system. All the results are expressed in terms of coupled linear matrix inequalities, which are solvable by optimization tools and directly reducible to the conditions of the integer order nonlinear switching systems as well as the conventional non-switched nonlinear fractional order systems. The proposed method has various practical implications. As an example, it is utilized to control Francis hydro-turbine governing system. This system is represented as a switching structure and supposed to supply a load suffering abrupt changes driven by an arbitrary switching mechanism. The simulation results support the usefulness of the method.

KEYWORDS: Fractional Order System, Switched Dynamical System, Lyapunov Theory, Stabilization, Linear Matrix Inequality, Hydro-Turbine Governing System.

1. Introduction

Switching systems are a class of hybrid systems that has attracted increasing attentions during the past decades. Various practical structures such as mechanical systems, power systems, networked controlled systems and multi agent systems could be described by switching dynamical systems [27]. This is due to the nature of switching systems in expressing the interactions between the continuous variable dynamics and the discrete events through multiple subsystems governed by a switching mechanism. These systems help to model the structural variations induced by external or internal discrete events such as failures, environmental factors, and configuration conversions [37, 38].

The stability, stabilization and control of switched systems are fundamental and interested research problems since they cannot be directly deduced from the specifications of each individual subsystem. A wide range of constructive researches is now available on the stability [19, 38, 41], stabilization and control [16-18], estimation and filtering [14, 47] of switched systems. In this area, the problems of nonlinear switched systems constitute a more significant focus since most real systems are essentially nonlinear but complex and hardly handled. Problems of nonlinear switched systems are addressed in [33, 38, 45, 49] and the references therein. Typically, the approach adopted to analyse these systems is utilizing the theories developed for nonlinear differential equations as well as the Lyapunov stability theory.

In all of the previous studies dedicated to the switched dynamical systems, the subsystems are supposed to be represented by the conventional integer order differential equations. It is already known that fractional calculus, as a generalization to the classical integer order calculus, provides a much better understanding of the realistic applications such as in electronic circuits [1, 21], electrical machines [35] and chemical systems [20]. Fundamentals of fractional order systems (FOSs) are pretty well established by now [4, 8, 31]. Various subtle results on the stability problems are discussed in [10, 26, 39]. Controllers are designed in [23, 24, 29] and observer-based designs are proposed in [2, 11]. Further reviews are also reported in [31, 25].

An increased number of applications of FOSs in various areas of science and technology as well as the potentials of switching systems to model structure-varying systems necessitates the analysis of fractional order switching systems (FOSSs). There exist only few studies dedicated to FOSSs. Basic stability notions of FOSS are studied in [22, 46]. Stabilization and control problems of linear FOSSs are mentioned in [3, 5, 6, 29]. Also, special classes of fractional order positive switched systems and fractional order impulsive switched system are studied in [28, 48] and [13, 44, 48], respectively.

Despite the recent developments on switching systems and FOSs, the control problems of fractional order switched dynamical structures have not received enough attention. These systems have different features compared with ordinary integer order systems and studying their problems is more challenging than both integer order and switched systems. Accordingly, the contribution of this paper is to investigate the stabilizability and stabilization of such systems.

The main contribution of this paper is to study the stabilizability and controller design of a class of nonlinear continuous-time dynamical systems under arbitrary switching. The system under consideration is supposed to include a linear nominal part with an unknown nonlinearity of Lipschitz type. This class is interested since it can describe real physical systems more precise and direct than the conventional linear, non-switched or integer order systems. The problem is solved through introducing an equivalent switched frequency distributed model for the system. Based on the equivalent model, the stabilizability of the system is investigated based on the stability definitions of switched systems and fractional order systems. The result is the stabilizability condition obtained by the Lyapunov approach, as well as the stabilizing controller gains designed in terms of linear matrix inequalities (LMIs). The results obtained for FOSSs in this paper are reducible to the linear fractional order switching systems, the integer order switching systems and the conventional fractional order systems reported in the previous studies. As an example, the method is tested on Francis hydro-turbine governing power system with fractional order dynamics subject to nonlinearities and time-varying load profile which is modelled by a switched structure. The results support the usefulness of the method.

The remainder of the paper is organized as follows: in Section 2, the dynamical specifications of the FOSSs are formulated and some preliminaries are recalled for both switching and fractional order systems. In Section 3, a stabilizability condition is derived and the controller is synthesized. In Section 4, the theoretical results are tested on the practical system. Finally, concluding remarks and possible future study directions are mentioned in Section 5.

2. Preliminaries and Problem Formulation

The fractional order nonlinear switched system (FONSS) is described as follows:

$$\begin{cases} D^{\alpha}x(t) = \mathbb{F}(x(t), u(t), r_t) \\ \qquad\quad = A(r_t)x(t) + f(x(t), r_t) + B(r_t)u(t), \\ x(t_0) = x_0,\ r_{t_0} = r_0 \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector of the system, $u(t) \in \mathbb{R}^m$ is the input vector of the system. $\{r_t, t \geq 0\}$ is a continuous-time switching mechanism [15] taking values in the finite set $\underline{\mathrm{N}} = \{1, 2, \ldots, N\}$, where the set $\underline{\mathrm{N}}$ contains the modes of the system and N is their number. The switching mechanism is of arbitrary type with no specific limitations or constraints.

$f(x(t), r_t)$ is a nonlinear dynamic related to $x(t)$ and r_t. $A(r_t)$ is the mode-dependent system matrix with compatible dimensions representing the nominal part of the system and $B(r_t)$ is a mode dependent system gain on the input $u(t)$. Also, x_0 is the initial state vector and r_0 represents the initial mode.

D^{α} denotes the fractional order operator representing both differential and integral operations. This operator is expressed by (2) according to the Caputo fractional derivative of the αth ($\alpha > 0$) order [9],

$$D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{h^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \tag{2}$$

where n is the nearest integer bigger than α, *i.e.* n is an integer satisfying n-1 < $\alpha \leq n$. $h^{(n)}(.)$ is the nth derivative of the function $h(.)$ and $\Gamma(.)$ is the Gamma function defined by $\Gamma(z) = \int_0^t e^{-z}t^{z-1}dt$. Remarkably, $0 < \alpha < 1$ is the fractional commensurate order of the system.

Remark 1: Hereafter in the paper, for the convenience of notations, $r_t = i$ is used and the subsystem matrices are labeled by A_i, B_i, $f_i(x(t))$ and $\mathbb{F}_i(x(t), u(t))$.

Remark 2: The FONSS of (1) is a general representation that can be used to model many physical systems in real world applications. The fractional differential equations enhance the representation precision while, the nominal part of (1) describes the linearization of the systems. Also, the nonlinear term represents various errors and uncertainties such as linearization errors or the external disturbances. Furthermore, the signal of r_t justifies multiple operating points of the system each described by a distinct subsystem.

Assumption 1: The vector field of the system (1), $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous and differentiable at the origin; and therefore $\mathbb{F}_i(0,0) = 0$.

Assumption 2: $f(x(t), r_t)$ is a continuous nonlinear function which is Lipschitz in x with the Lipschitz constant $\gamma_i > 0$, *i.e.*,

$$\|f_i(\hat{x}(t)) - f_i(x(t))\| \leq \gamma_i \|\hat{x}(t) - x(t)\| \tag{3}$$

for all $\hat{x}(t), x(t) \in \mathbf{R}^n$ and $f_i(0)=0$.

Before proceeding further, the following lemmas, definition and proposition, which will be used in the derivation of the main results, are recalled.

Lemma 1: (Continuous frequency distributed model) [40]. The fractional order nonlinear differential equation $D^{\alpha}x(t) = g(x(t)$ can be expressed as (4) due to the continuous frequency distribute model of the fractional integrator,

$$\begin{cases} \dfrac{\partial Z(\omega, t)}{\partial t} = -\omega Z(\omega, t) + g(x(t)) \\ x(t) = \int_0^{\infty} \mu(\omega) Z(\omega, t) d\omega, \end{cases} \tag{4}$$

where $Z(\omega,t)$ is an auxiliary time and frequency domain function and $\mu(\omega)$ is introduced as

$$\mu(\omega) = \frac{\sin(\alpha\pi)}{\pi}\omega^{-\alpha}. \tag{5}$$

Lemma 2: (Schur complement) [7]. For a given matrix $S = S^T$, the following assertions are equivalent:

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} < 0 \quad (6)$$

$$S_{11} < 0, S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0 \quad (7)$$

$$S_{22} < 0, S_{11} - S_{12}^T S_{22}^{-1} S_{12} < 0. \quad (8)$$

Lemma 3: [42] Let Y be a symmetric matrix, H and E be given matrices with the appropriate dimensions. For $F(t)$ satisfying $F(t)^T F(t) \le I$, the inequality (9) holds

$$HFE + E^T F^T H^T < 0 \quad (9)$$

if and only if there exists an $\varepsilon > 0$ such that

$$HFE + E^T F^T H^T \le \varepsilon HH^T + \varepsilon^{-1} E^T E. \quad (10)$$

Definition 1: (Common Lyapunov Function) [38]. A function $V(x(t))$ is said to be a common (strong) Lyapunov function for the conventional switching system $\dot{x}(t) = g(x(t), r_t)$ if:

1. It is continuous everywhere and continuously differentiable except possibly at the origin.
2. It admits class K_∞ bounds, *i.e.* there are class K_∞ functions of β_1 and β_2 such that, $\beta_1(|x(t)|) \le V(x(t)) \le \beta_2(|x(t)|)$.
3. There is a class K function β_3 such that $\dot{V}(x(t)) \le -\beta_3(|x(t)|)$.

Proposition 1: (Stability) [38]. The switched system $\dot{x}(t) = g(x(t), r_t)$ is uniformly asymptotically stable if it admits a common Lyapunov function.

By introducing the FONSSs as a generalization of both switching and fractional order nonlinear systems, now, the stabilizability and stabilization problems can be addressed.

3. Main Results

In this section, the stabilization problem of nonlinear fractional order switched dynamical system is investigated. To this end, first, the continuous mode-dependent frequency distributed model is obtained for the system. Subsequently, Lyapunov approach is utilized to give the condition. All the results are reported in the LMI form and can be checked easily.

Consider the state feedback controller of the following form:

$$u(t) = K(r_t)x(t), \quad (11)$$

where $K(r_t)$ is the mode dependent gain to be designed.

Applying the controller gains of (11) to the system of (1) yields the closed-loop system as (12),

$$\begin{cases} D^\alpha x(t) = \overline{A}(r_t)x(t) + f(x(t), r_t) \\ x(t_0) = x_0, \; r_{t_0} = r_0, \end{cases} \quad (12)$$

where

$$\overline{A}(r_t) = A(r_t) + B(r_t)K(r_t). \quad (13)$$

By using the Lyapunov approach, first the sufficient condition for stability of the closed-loop dynamic system is obtained.

Theorem: The FONSS of (1) with the commensurate order α, $0 < \alpha < 1$ is stabilizable with the controller gains (11) if, there are symmetric, positive definite matrix X, matrices Y_i together with the real scalars $\varepsilon_i > 0$ and $\gamma_i > 0$ such that the following set of LMIs hold:

$$\begin{bmatrix} J_i & \gamma_i X \\ \gamma_i X & \varepsilon_i I \end{bmatrix} < 0, \quad (14)$$

where

$$J_i = XA_i^T + XA_i + Y_i^T B_i^T + B_i Y_i + \varepsilon_i I. \quad (15)$$

Then, the controller gains are obtained as $K_i = Y_i X^{-1}$.

Proof: Following Lemma 1, the closed-loop dynamic system of (12) can be written as:

$$\begin{cases} \dfrac{\partial Z(\omega, t)}{\partial t} = -\omega Z(\omega, t) + \overline{A}(r_t)x(t) + f(x(t), r_t) \\ x(t) = \int_0^\infty \mu(\omega) Z(\omega, t) d\omega. \end{cases} \quad (16)$$

Based on the continuous frequency distributed system (16), consider the two Lyapunov functions:

$v(\omega, t)$ as the Lyapunov function corresponding to the elementary frequency ω and $V(t)$ defined by (17) as the Lyapunov function summing all the $v(\omega, t)$ with the weighting function $\mu(\omega)$. Here, P denounces a symmetric and positive definite matrix.

$$V(t)=\int_0^\infty \mu(\omega)v(\omega,t)d\omega=\int_0^\infty \mu(\omega)\,Z^T(\omega,t)PZ(\omega,t)d\omega. \tag{17}$$

The derivative of the Lyapunov function along the solution trajectories of (16) becomes as (18)

$$\dot V(Z(t))=\int_0^\infty \mu(\omega)\,\dot Z^T(\omega,t)PZ(\omega,t)d\omega+ \int_0^\infty \mu(\omega)Z^T(\omega,t)P\dot Z(\omega,t)d\omega. \tag{18}$$

Substituting the equations of (16) into (18) provides:

$$\dot V(Z(t))=\int_0^\infty\Big(\mu(\omega)\big\{-\omega Z(\omega,t)+\overline{A}_i x(t)+f_i(x(t))\big\}^T PZ(\omega,t)\Big)d\omega+ \int_0^\infty\Big(\mu(\omega)Z^T(\omega,t)P\big\{-\omega Z(\omega,t)+\overline{A}_i x(t)+f_i(x(t))\big\}\Big)d\omega. \tag{19}$$

Simplifications yield:

$$\begin{aligned}\dot V(Z(t))=&-2\int_0^\infty \omega\mu(\omega)Z^T(\omega,t)PZ(\omega,t)d\omega+\\ &\int_0^\infty \mu(\omega)\,x^T(t)\overline{A}_i^T PZ(\omega,t)d\omega+\\ &\int_0^\infty \mu(\omega)\,Z^T(\omega,t)P\overline{A}_i x(t)d\omega+\\ &\int_0^\infty \mu(\omega)\,f_i^T(x(t))PZ(\omega,t)d\omega+\\ &\int_0^\infty \mu(\omega)\,Z^T(\omega,t)Pf_i(x(t))d\omega\end{aligned} \tag{20}$$

which turns to the following using (16),

$$\begin{aligned}\dot V(Z(t))=&-2\int_0^\infty \omega\mu(\omega)Z^T(\omega,t)PZ(\omega,t)d\omega+\\ &x^T(t)\overline{A}_i^T Px(t)+x^T(t)P\overline{A}_i x(t)+\\ &f_i^T(x(t))Px(t)+x^T(\omega,t)Pf_i(x(t))\end{aligned} \tag{21}$$

The first item in the right-hand side of (21) is less than zero, therefore the inequality (22) holds

$$\begin{aligned}&-2\int_0^\infty \omega\mu(\omega)Z^T(\omega,t)PZ(\omega,t)d\omega+x^T(t)\overline{A}_i^T Px(t)+\\ &x^T(t)P\overline{A}_i x(t)+f_i^T(x(t))Px(t)+x^T(t)Pf_i(x(t))\le\\ &x^T(t)\overline{A}_i^T Px(t)+x^T(t)P\overline{A}_i x(t)+f_i^T(x(t))Px(t)+\\ &x^T(t)Pf_i(x(t))\end{aligned} \tag{22}$$

Subsequently, if the inequality (23) holds for $\forall i\in \underline{N}$,

$$x^T(t)\overline{A}_i^T Px(t)+x^T(t)P\overline{A}_i x(t)+f_i^T(x(t))Px(t)+ x^T(t)Pf_i(x(t))<0 \tag{23}$$

one has $\dot V(Z(t))<0$. Substituting (13) in (23) leads to (24)

$$x^T(t)\big(A_i+B_iK_i\big)^T Px(t)+x^T(t)P\big(A_i+B_iK_i\big)x(t)+ f_i^T(x(t))Px(t)+x^T(t)Pf_i(x(t))<0. \tag{24}$$

Remarkably, (24) can be rewritten as (25) utilizing the following inequality obtained by Lemma 3

$$f_i^T(x(t))Px(t)+x^T(t)Pf_i(x(t))\le \varepsilon_i x^T(t)P^2x(t)+\varepsilon_i^{-1}\gamma_i^2\,x^T(t)x(t). \tag{25}$$

Utilizing (25), the inequality of (24) turns to (26):

$$x^T(t)\big(A_i+B_iK_i\big)^T Px(t)+x^T(t)P\big(A_i+B_iK_i\big)x(t)+ \varepsilon_i\,x^T(t)P^2x(t)+\gamma_i^2\varepsilon_i^{-1}x^T(t)x(t)<0, \tag{26}$$

which is equivalent to (27) due to its symmetric form:

$$\big(A_i+B_iK_i\big)^T P+P\big(A_i+B_iK_i\big)+\varepsilon_i P^2+\varepsilon_i^{-1}\gamma_i^2<0. \tag{27}$$

Considering the quadratic and bounded Lyapunov function of (17) along with its negative definite form guaranteed by (27), the closed-loop system of (12) is stabilizable according to Definition 1 and Proposition 1.

Notably, the condition of (27) is nonlinear in P and K_i. To find the controller gains, it is desired to transform (27) into an LMI form, so let $X = P^{-1}$. Pre- and post-multiplying (27) by X yields (28).

$$XA_i^T + XA_i + XK_i^T B_i^T + B_i K_i X + \varepsilon_i I + \varepsilon_i^{-1}\gamma_i^2 X^2 < 0, \quad (28)$$

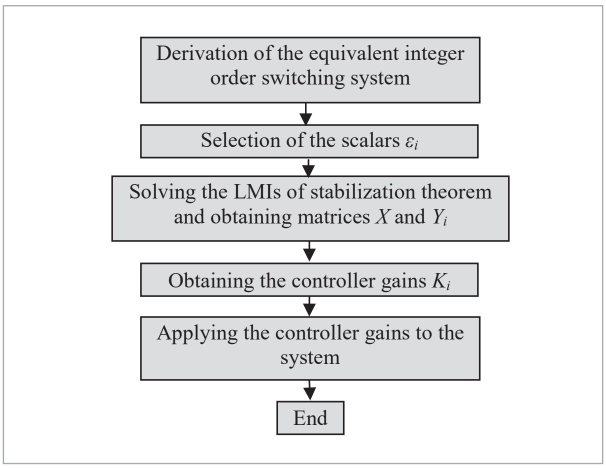
Define $Y_i = K_i X$. Therefore, (27) changes to (29):

$$XA_i^T + XA_i + Y_i^T B_i^T + B_i Y_i + \varepsilon_i I + \varepsilon_i^{-1}\gamma_i^2 X^2 < 0, \quad (29)$$

Using Lemma 2 makes it possible to write (29) in the form of (14) and (15). Finally, the state feedback gains are derived as $K_i = Y_i X^{-1}$, which ends the proof.

The flowchart of the proposed method is shown in Figure 1.

Figure 1
The flowchart of the proposed method



Remark 3: The stability criterion obtained for the nonlinear fractional order switched system is directly reducible to the results which have been previously reported for the integer order switched systems [38], and the ordinary fractional order systems [32, 36].

4. Practical Application

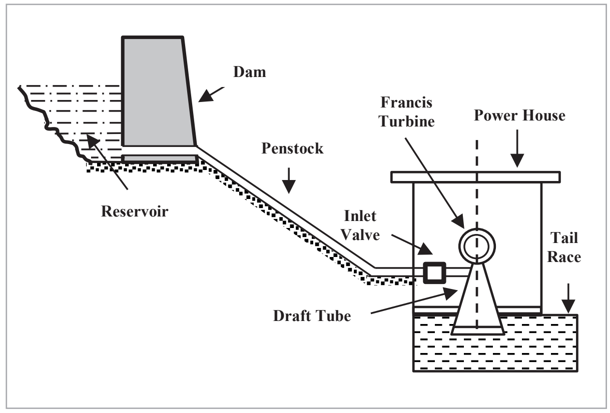
In this section, the simulation results are provided for a nonlinear fractional order hydro-turbine governing system (HGS) [43] to show the merits of the proposed method.

The fractional order Francis hydro-turbine governing system is composed of a hydro-turbine and penstock system, a generator system and a hydraulic servo system. The system is modelled as a switching structure which is supposed to supply a load that suffers abrupt changes driven by an arbitrary switching mechanism.

4.1. Modelling of the Switched Nonlinear Francis Hydro-Turbine Governing System

The basic physical model of the hydro power penstock system is shown in Figure 2.

Figure 2
Schematic of Francis hydro Turbine Governing System



The dynamic characteristic of the synchronous generator is given by:

$$\begin{aligned} \dot{\delta}(t) &= \omega_0 \omega(t) \\ \dot{\omega}(t) &= \frac{1}{T_a + T_b}\left(-D\omega(t) + m_t(t) - m_e(t)\right), \end{aligned} \quad (30)$$

where $\delta(t)$ is the rotor angle, $\omega(t)$ is the variation of the generator speed and ω_0 is the rated angular speed of the generator. Additionally, D is the damping factor of the generator and it is generally regarded as a constant. $m_t(t)$ is the output torque of the hydro turbine while $m_e(t)$ denotes the torque of the electrical load. T_a is the inertia time constant of the generator and T_b denotes the inertia time constant of the load.

If the influence of the rotor on the torque is added to the damping factor, the torque of the electric load and the terminal active power are equal to each other, *i.e.*

$$P_e(t) = m_e(t), \quad (31)$$

The electromagnetic power of the generator can be described as (32)

$$P_e(t)=\frac{E_q' V_s}{x_d'}\sin\delta(t)+\frac{V_s^2}{2}\frac{x_d'-x_q}{x_d' x_q}\sin 2\delta(t), \tag{32}$$

where E_q' is the transient internal voltage of the armature, V_s is the voltage of bus at infinity, x_d' and x_q are the direct axis and quartered transient reactances, respectively.

Dynamic characteristic of a hydraulic servo system is given as (33) according to the fractional calculus

$$D^\alpha y(t)=\frac{1}{T_y}\big(u(t)-y(t)\big), \tag{33}$$

where $y(t)$ is the incremental deviation of the guide vane opening and T_y is the major relay connector response time. Here, $u(t)$ is the output signal of the governing system, which is the input voltage for the electric-hydraulic servo system.

The output torque of the turbine governing system is obtained as

$$\dot{m}_t(t)=\frac{1}{e_{qh}T_\omega}\left(-m_t(t)+e_y y(t)+\frac{ee_yT_\omega}{T_y}y(t)\right). \tag{34}$$

In (34), e_{qh} is the transfer coefficient of turbine flow on the head, T_ω is the water inertia time constant of the penstock system, e_y is the transfer coefficient of turbine torque flow on the servo motor stroke, e is defined as $e = e_{qh}\, e_h/e_y - e_{qh}$, and e_h is the transfer coefficient of turbine torque on the water head.

It is supposed that the system load is varying according to an arbitrary switching mechanism.

The load is assumed to be characterized by two distinguished inertia time constants characterized by $T_b(r_t)$:

$$T_b(r_t)=\begin{cases}T_{b1} & r_t=1\\ T_{b2} & r_t=2,\end{cases} \tag{35}$$

where T_{b1} refers to load with large time constant while T_{b2} specifies a load with small time constant.

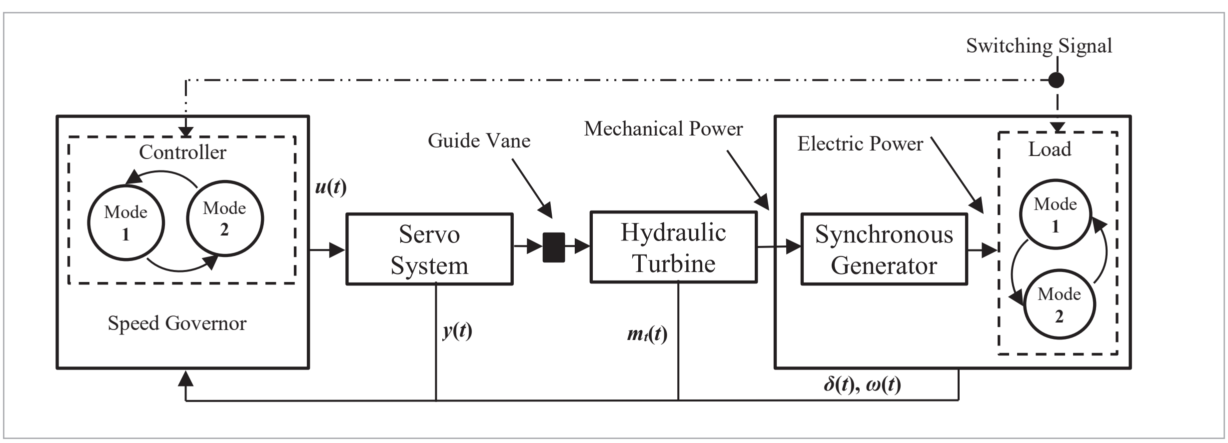
The block diagram of the proposed control system is depicted in Figure 3. The system is composed of a switching controller whose modes change according to the same switching mechanism that orchestrates the loads.

Considering the Equations (30) to (35), the mathematical switching model of the hydro-turbine governor system is obtained as (36) where the fractional order is $\alpha = 0.98$:

$$\begin{cases}\dot{\delta}(t)=\omega_0\omega(t)\\ \dot{\omega}(t)=\dfrac{1}{T_a+T_b(r_t)}\left(-D\omega(t)+m_t(t)-\dfrac{E_q'V_s}{x_{d\Sigma}'}\sin\delta(t)\right.\\ \qquad\left.-\dfrac{V_s^2}{2}\dfrac{x_d'-x_q}{x_d'x_q}\sin 2\delta(t)\right)\\ \dot{m}_t(t)=\dfrac{1}{e_{qh}T_\omega}\left(-m_t(t)+e_y y(t)+\dfrac{ee_yT_\omega}{T_y}y(t)\right)\\ D^\alpha y(t)=\dfrac{1}{T_y}\big(u(t)-y(t)\big).\end{cases} \tag{36}$$

Figure 3

The proposed control structure for the Francis hydro-turbine governing system



To proceed further, the parameters of system (35) are selected as shown in Table 1. Remarkably, the state variables as well as the parameters are in *p.u.* while the time constants are in *seconds*.

Table 1

Parameter values of the HGS system

Parameter	Value	Parameter	Value
D	2	ω_0	300
T_ω	0.8	T_y	0.1
T_a	10	T_{b1}	9
T_{b2}	0.6	V_s	1
x_q	1.474	x_q'	1.25
e	0.7	e_{qh}	0.5
e_y	1	-	-

Defining the state variables of the HGS as $x_1(t) = \delta(t)$, $x_2(t) = \omega(t)$, $x_3(t) = m_t(t)$, $x_4(t) = y(t)$ and replacing the parameter values, reveals the final form of the system model as (37)

$$\begin{cases} \dot{x}_1(t) = 300x_2(t) \\ \dot{x}_2(t) = \dfrac{1}{10+T_b(r_t)}\big(-2x_2(t)+x_3(t)-1.08\sin x_1(t) \\ \qquad -0.061\sin 2x_1(t)\big) \\ \dot{x}_3(t) = -2.5x_3(t)+6.6x_4(t) \\ D^{0.98}x_4(t) = -10x_4(t)+10u(t). \end{cases} \tag{37}$$

Accordingly, the system matrices in the form of (1) are obtained as (38):

$$A_1 = \begin{bmatrix} 0 & 300 & 0 & 0 \\ 0 & \frac{-2}{19} & \frac{1}{19} & 0 \\ 0 & 0 & -2.5 & 6.6 \\ 0 & 0 & 0 & -10 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 300 & 0 & 0 \\ 0 & \frac{-2}{10.6} & \frac{1}{10.6} & 0 \\ 0 & 0 & -2.5 & 6.6 \\ 0 & 0 & 0 & -10 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 0 & 0 & 0 & 10 \end{bmatrix}^T$$
$$f_1(x(t)) = \begin{bmatrix} 0 & \frac{-1.08}{19}\sin x_1(t) + \frac{-0.061}{19}\sin 2x_1(t) & 0 & 0 \end{bmatrix}^T$$
$$f_2(x(t)) = \begin{bmatrix} 0 & \frac{-1.08}{10.6}\sin x_1(t) + \frac{-0.061}{10.6}\sin 2x_1(t) & 0 & 0 \end{bmatrix}^T. \tag{38}$$

4.2. Simulation Results of the Switched Nonlinear Francis hydro-Turbine Governing System

Uncontrolled states of the fractional order HGS under two different arbitrary switching signals shown by Figure 4 are illustrated in Figure 5. As the figure shows, the uncontrolled system is unstable.

Figure 4

Two sample arbitrary switching signals (SW1, SW2)

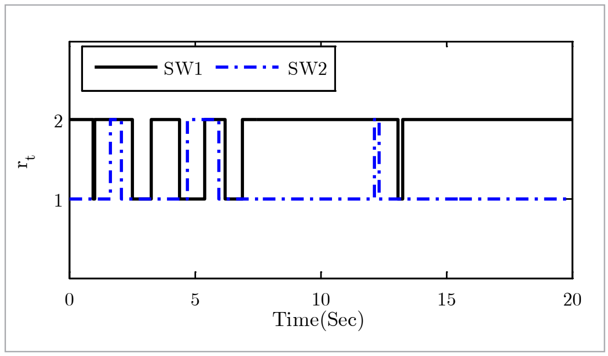
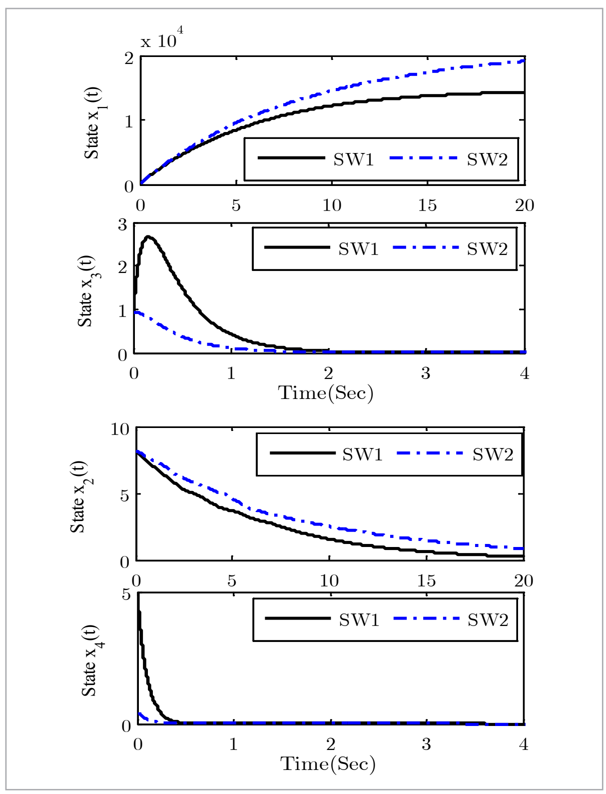


Figure 5

Uncontrolled states of HGS under two arbitrary switching signals (SW1, SW2)



Solving the LMIs of Theorem provides the matrices and controller gains as (39). The parameters are selected as $\gamma_1 = \gamma_2 =$ 10e-10, $\varepsilon_1 = \varepsilon_1 =$ 10e-7.

The resultant control outputs of the HGS appear in Figure 6. The controlled states of the HGS are shown in Figure 7. Clearly, the states are stabilized properly.

Figure 6

Controller outputs under two arbitrary switching signals

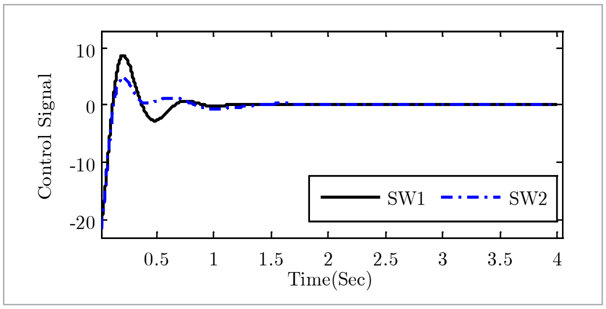
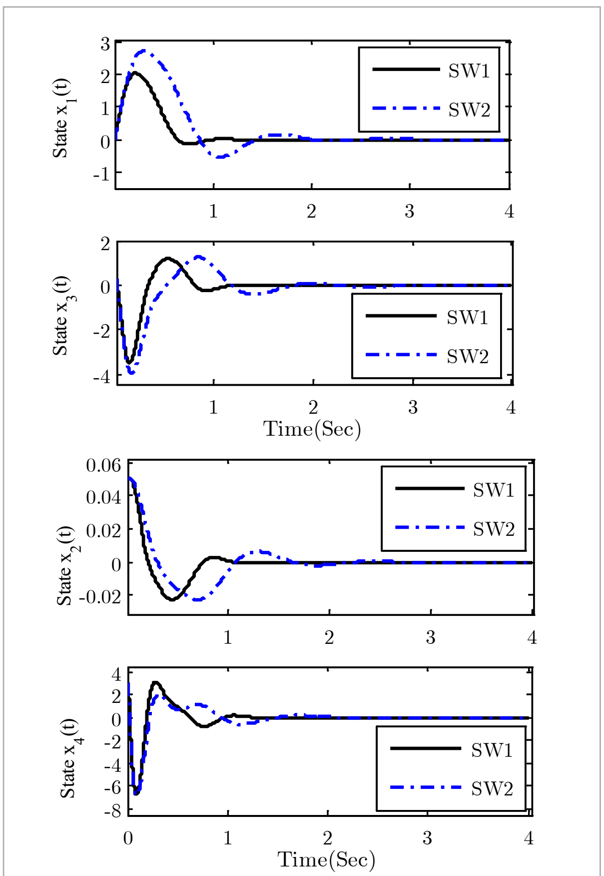


Figure 7

Controlled states of HGS under two arbitrary switching signals (SW)



$$X = \begin{bmatrix} 0.2237 & -0.0017 & -0.0025 & 0.0421 \\ -0.0017 & -0.0003 & -0.0014 & -0.0011 \\ -0.0025 & -0.00145 & 0.1651 & -0.0198 \\ 0.0421 & -0.0011 & -0.0201 & 0.2504 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.0611 & -0.0008 & -0.1839 & 0.1802 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.0548 & -0.0008 & -0.1808 & 0.1811 \end{bmatrix} \tag{39}$$

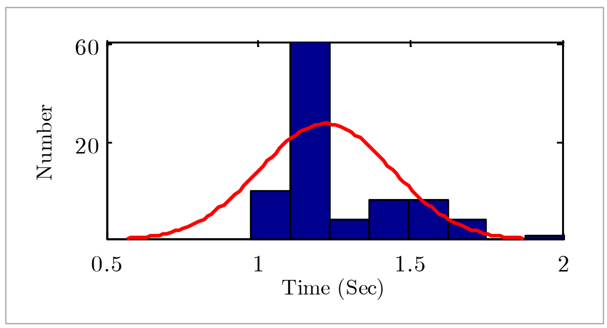
$$K_1 = \begin{bmatrix} -2.4054 & -364.723 & -4.4487 & -0.8520 \end{bmatrix}^T,$$

$$K_2 = \begin{bmatrix} -2.3584 & -353.8254 & -4.3276 & -0.8024 \end{bmatrix}^T.$$

To further investigate the proposed theorem, the simulations are conducted fot 100 different switching signals. The statistics of the settling time and its approximated distribution are summarized in Figure 8. The figure shows that the convergence time has a normal distribution with mean value of 1.2217 seconds and the deviation of 0.2174 seconds.

Figure 8

Statistics of the settling time of the controlled states for 100 different arbitrary signals



Remarkably, a drawback of the proposed method is that the conditions depend on a number of parameters ε and γ that must be suitably tuned. The parameter ε rises due to Lemma 3 used for dealing with the system nonlinearity. According to [42], this lemma holds for any ε > 0. Although this parameter could take any values, it is preferred to be selected properly. The reason is that this parameter determines the robustness degree of the system and improper values may increase conservativeness and even lead to infeasible LMI sets.

To investigate the effect of ε on the responses, the controlled system states and control signals are depicted in Figures 9-10 under the switching of Figure 11.

Figure 9

The effect of the parameter ε on the controlled states of HGS

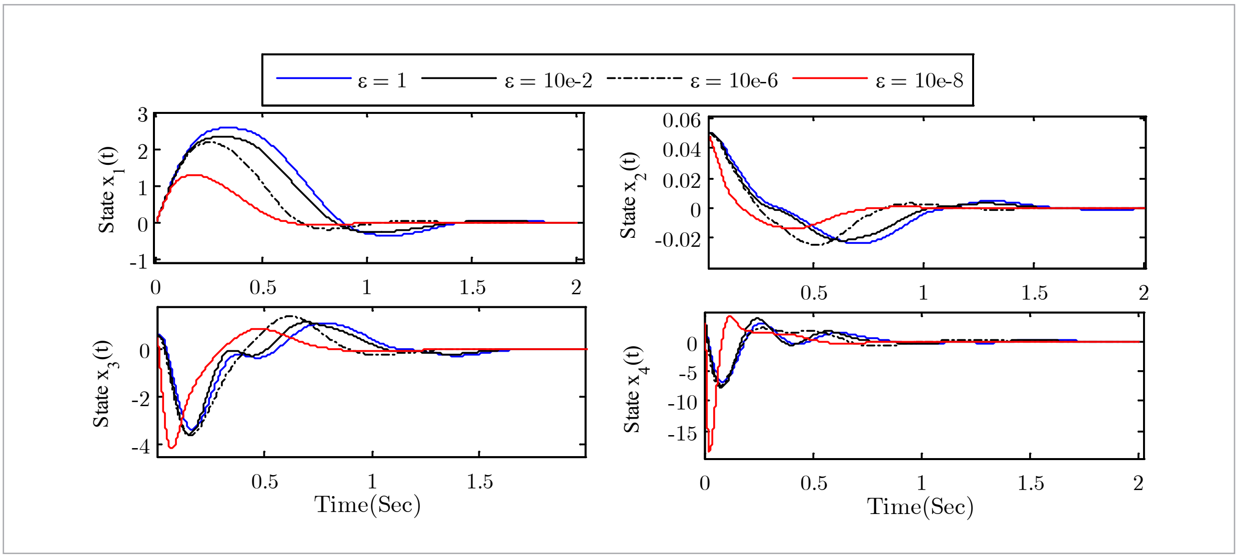


Figure 10

The effect of ε on the control signal variation

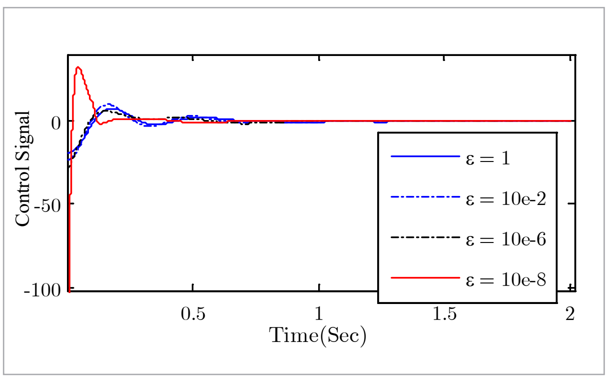
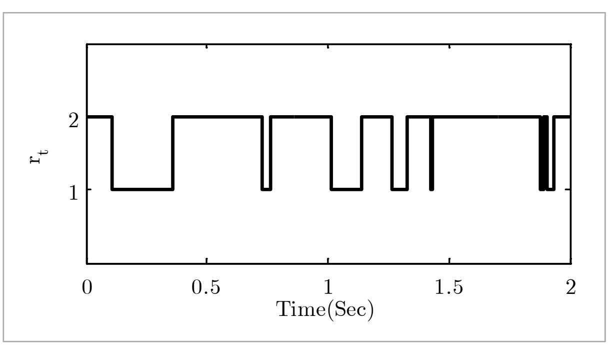


Figure 11

Sample arbitrary switching signals for ε analysis for the load variation



Similarly, the parameter of γ appears as a result of Assumption 2, it is a positive value that fulfils the Lipschitz condition. The effect of this parameter is also simulated in Figures 12-14.

As the figures show, smaller values of both parameters ε and γ lead to faster and less conservative responses.

At the same time, they provide more aggressive and larger controller outputs. Since the variations of the controller output and the conservativeness of the responses are conflicted specifications, proper selection of both parameters is required.

There exist two approaches for dealing with the parameters ε and γ. The first approach is to select them a priori to afford a prescribed degree of robustness. This approach is extensively used in the controller design problems [44] and is also preferred in the current study. The main advantage of this approach is providing the conditions of a fair comparison between the multiple results. The second approach is to optimize those parameters which is addressed in [34].

It is worth mentioning that, for analysis, the parameters of ε and γ are selected the same for both modes without loss of generalization. Additionally, to provide a fair comparison between the results of distinct values of ε and γ, for the former γ is uniformly selected as γ = 10e-9, while for the latter ε is uniformly selected as ε = 10e-8 for both modes.

Figure 12

Sample arbitrary switching signals for γ analysis for the load variation

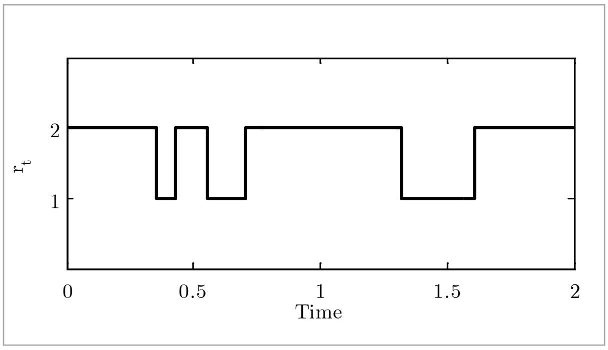


Figure 13

The effect of γ on the control signal variation

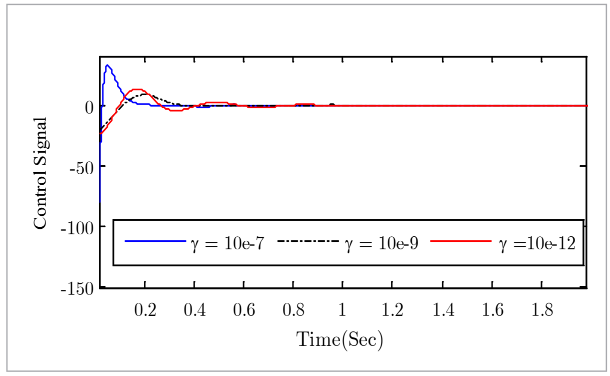
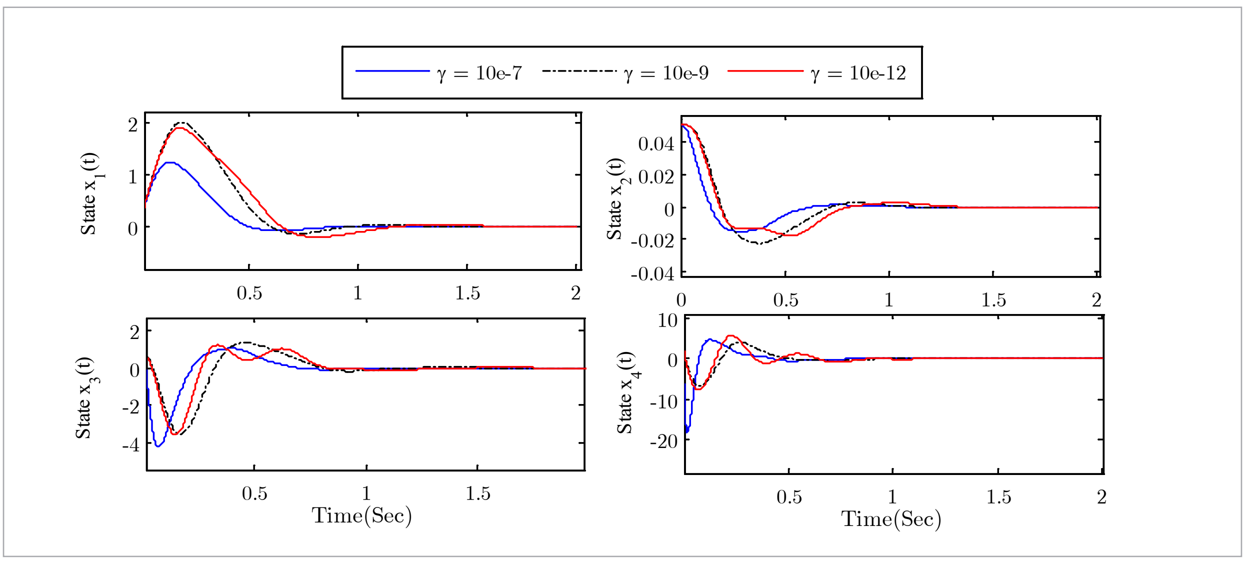


Figure 14

The effect of the parameter γ on the controlled states of HGS



5. Conclusions

In this paper, the stabilizability problem of nonlinear fractional order systems under arbitrary switching is addressed. The analysis is started by means of a frequency distributed mode-dependent equivalent model for the system. Subsequently, a condition guaranteeing the existence and the synthesis of multi-mode feedback stabilizing controller in LMI formulation is given. The computed controller law ensures the stabilization of the system under the nonlinear perturbations and the dynamical variations induced by the switching. The obtained theorem is a general form of the results previously reported for non-switched fractional order systems as well as the ordinary integer order switched systems. The proposed analysis and synthesis method is successfully tested on a hydro-turbine governing system modelled as a switching structure. Remarkably, further studies are required on the nonlinear fractional order dynamical systems under constrained switching mechanisms and their applications.

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